H a n s - J ü r g e n   B r o c k m a n n

# Coherent Mathematical Treatment of the Mutual Inductive and Resistive Coupling Acting in Electromagnetic Transformers

## Coherent measurements of the frequency response of the mutual resistance and the mutual inductance caused by the mutual Proximity-Effect

## Derivation of the universal formula for the determination of the maximum possible power transfer efficiency:

$$\eta_{max} = \frac{k_{ind}^2 \cdot Q_1 \cdot Q_2 + k_{prox}^2}{\left(\sqrt{1 + k_{ind}^2 \cdot Q_1 \cdot Q_2} + \sqrt{1 - k_{prox}^2}\right)^2}$$



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The here reproduced "Abstractum" is a revised version of the original "Abstract".

In comparison with the original title from 2014, the above given main title contains now additionally the word "Mutual", which is already frequently used in the original text. And the first subtitle is made more precise by adding the words "Coherent" and "mutual inductance", while the second subtitle received the technical term "power transfer". Furthermore I exchanged the adjective "elementary" against "universal", because of the general derivation in Chapter 10 and because of the verification for other networks than just "only" a transformer, as described on the second page of the Abstractum.

# Abstractum

with foundation of the two separate coupling coefficients $k_{ind}$ and $k_{prox}$ :

Assuming a loss-free transformer, of which primary and secondary coil have the same inductances at not neccessarily identical turn numbers, the coupling coefficient is equal to the ratio of the no-load output voltage to the input voltage. But at different inductances we have to consider the transformer not only in the forward direction but also in the backward direction, and finally we have to form the geometrical mean of the inductances. Then according to page 22 of Chapter 1 is :

$$k_{ind} = \sqrt{\frac{C}{A}\cdot\frac{B}{D}} = \sqrt{\frac{M}{L_1}\cdot\frac{M}{L_2}} = \frac{M}{\sqrt{L_1\cdot L_2}} \qquad = k_i \text{ , a shortened notation only.}$$

Analogously, the following holds for a pure **r**esistive four-terminal network, called also two-port network:

$$k_{prox} = \sqrt{\frac{C}{A}\cdot\frac{B}{D}} = \sqrt{\frac{R_m}{R_1}\cdot\frac{R_m}{R_2}} = \frac{R_m}{\sqrt{R_1\cdot R_2}} \qquad = k_r \text{ , a shortened notation only.}$$

These both coupling coefficients are real numbers. But if we accept and combine the coherent character of both equations of the four trminal network, we obtain the complex coupling coefficient:

$$k = \sqrt{\frac{C}{A}\cdot\frac{B}{D}} = \sqrt{\frac{(R_m + j\cdot\omega\cdot M)}{(R_1 + j\cdot\omega\cdot L_1)}\cdot\frac{(R_m + j\cdot\omega\cdot M)}{(R_2 + j\cdot\omega\cdot L_2)}} = \frac{R_m + j\cdot\omega\cdot M}{\sqrt{(R_1 + j\cdot\omega\cdot L_1)\cdot(R_2 + j\cdot\omega\cdot L_2)}}$$

Here I would like to cofess that during all my own work in the development laboratory including the optimization of the efficiency of small transformers, it was always quite common to use the directly measured values of $L_1$ , $R_1$ , $L_2$ , $R_2$ and both short circuit inductances, from which the inductive coupling coefficient has been approximately calculated according to $\sqrt{1 - \frac{L_{1sc}}{L1}} = \sqrt{1 - \frac{L_{2sc}}{L_2}}$ ( sc = short circuit )

But after the laboratory was equipped with an impedance analyser, I could determine all 4 complex coefficients of a four terminal network, based on the complex open-circuit and short-circuit impedance measurement values. That calculation I derived mathematically and is described here in Chapter 7. But already in the middle of Chapter 1 I applied polar coordinates. Other versions are from a later date after my retirement: Furthermore I derived formulae for the separate $k_{ind}$ and $k_{prox}$ as functions of $Q_1$ , $Q_2$ and the complex coupling coefficient $k = k_{real} + i\cdot k_{im}$ , and vice verta. In Chapter 5 , starting from unbalanced measurement values, but forming the complex geometric mean of $A$ and $D$ , I applied polar coordinates, and compared results with previous calculations, balanced via the created **C**omplex **R**elative-**E**rror **E**qualization **T**heory, the **CREET** of Chapter 2 . But in Chapter 6 I began the calculation of $R_m$ and $M$ with the complex geometrical mean of both short circuit impedances.

In Germany I had certainly derived already formulae for the mutual resistance $R_m$ and the mutual inductance $M$ . And finally later in Finland, thanks to the disposal of a precise impedance analyser in the laboratory I asked myself, what could $R_m$ and the resistive coupling coefficient $k_{prox}$ be used for ? But, if I remember correctly, it was not easy to get anyone really interested in my reasoned derivations. - For instance: Once I received from the friendly physics professor Dr. habil. Dr. h. c. mult. Hans-Jürgen Treder from Potsdam the following literature recommendation, intended to enrich my limited knowledge:

"BEHANDLUNG VON SCHWINGUNGSAUFGABEN
mit komplexen Amplituden und mit Vektoren", written by Prof. Dr. Hans Georg Möller, Hamburg, 2. Auflage 1936.

This book contains a lot of approximative measurements and calculations of transformers, descriptions of the Skin-Effect, eddy currents, resonance circuits, radio transmitters, three-phase asynchron motors, the theory of transmission lines and so on, but not just that treatment I have created later in my book, published in 2014.

But the hint to this book has become suddenly very important for me due to a certain personal reason, because already in the library of the university of Helsinki I found to my great surprise that all graphical representations and many mathematical formulae were already familiar to me, because I remembered an identical book, which I already during my school days had found in the bookcase of my fallen father. Unfortunately, later in West Germany never I was able to remember neither title nor author of this book. -

But I remembered exactly that in that special book of my father was not defined a resistive coupling coefficient. And I remembered also that in the last two years, before I graduated 1956 from the "Karl-Marx-Oberschule", I had already been looking for a formula for the maximum possible power transfer efficiency of a transformer, but had not found it in that book, forgotten in Malchin in Mecklenburg in a hurry. -
But here now is the longed-for $\eta_{max}$ -formula, derived ambitiously in Finland, 2 years after my retirement.

This formula is a function of only the 4 parameters: $Q_1 = \frac{\omega \cdot L_1}{R_1}$ , $Q_2 = \frac{\omega \cdot L_2}{R_2}$ , $k_{ind}$ and $k_{prox}$ .

$$\eta_{max} = \frac{k_{ind}^2 \cdot Q_1 \cdot Q_2 + k_{prox}^2}{\left(\sqrt{1 + k_{ind}^2 \cdot Q_1 \cdot Q_2} + \sqrt{1 - k_{prox}^2}\right)^2}$$

Applied to a certain frequency $f = \frac{\omega}{2 \cdot \pi}$

This formula is an easily to use universal function and is derived in all details in Chapter 10, which includes the determination of the optimum terminating capactance $C_{opt}$ and the optimum terminating load resistance $R_{opt}$ , both connected in parallel with the secondary coil.
But if someone would not have sufficient patience to follow all the individual steps of the derivation, then the 2 verification examples with Two-Port $\Pi$-Networks of Chapter 13 could also bring satisfaction.

According to my own experience, the following inequality $0 < k_{prox} < k_{ind} < 1$ applies to transformers.
But $k_{prox}$ of the just mentioned 2 two-port $\Pi$-networks has a negative sign and a large absolute value:
1st example: $k_{prox} = -0.999884$ $k_{ind} = 0.530183$ , 2nd example: $k_{prox} = -0.998952$ $k_{ind} = 1$
The **mathematically fascinating thing** is now that the formulae for $\eta_{max}$ , $R_{opt}$ and $C_{opt}$ , derived in Chapter 10 apply exactly in all these cases, and furthermore that the $\eta_{max}$ -function gives the evidence that at increasing $k_{prox}$ also $\eta_{max}$ is increasing at whatever values the 3 other variables $Q_1$ , $Q_2$ and $k_{ind}$ may have. A certain result derived on page 69 of my book is, that the terminating capacitance $C_{opt}$ is independent from any variation of the terminating load resistance.
And of course, all derived formulae can be applied in an ideal way, after the raw measurement values have been balanced by means of the formulae of the **C**omplex **R**elative-**E**rror **E**qualization **T**heory.

Finally I would like to add here **a third** but very simple **verification** for the universal formula given above. Let us consider a resistor voltage divider, which can be treated as a two-port $\Pi$-network, and ask for that optimum external load resistance $x$ , for which the ratio $\eta(x)$ of the output power to the input power becomes a maximum. This task can be solved in the well-known way by deriving $\eta(x)$ as a function of the horizontal resistor $R_h$ and the right-side vertical resistor $R_v$ of a $\Pi$-network and of the external load resistance $x$ : First derivation of $\eta(x)$ with respect to $x$ , then equal to zero, and so on and so forth:

The result is: $x = R_{opt} = R_v \cdot \sqrt{\frac{R_h}{R_h + R_v}}$ and consequently $\eta_{max} = \frac{R_v^2 \cdot R_{opt}}{(R_v + R_{opt}) \cdot \left[ R_h \cdot R_v + (R_h + R_v) \cdot R_{opt} \right]}$

Together with the found $R_{opt}$ , the ratio of the output voltage to the input voltage can be calculated and with that the just obtained power ratio $\eta(x)$ . **A gratifying result is, that it exactly matches the formula given above** with $Q_1 \cdot Q_2 = 0$ and $k_{prox} = \frac{R_v}{\sqrt{(R_h + R_v) \cdot R_v}}$ , derived from a comparison of both equations of the two-port $\Pi$-network with the corresponding equations of a transformer given in Chapter 1.
For such a comparison it is useful to connect a negligible very small inductance $L_v$ in series with $R_v$ .

In Chapter 14 is described a method for the determination of short-circuit impedances without direct short-circuit measurements, but in place of that with measurements at practically used load resistors. The short-circuit impedance is then the result of an also derived algebraic extrapolation.

In Chapter 15 is given a program suitable for instance for the non-complex BASIC-system.
A lot of the most important detailed calculations of this work are carried out with high precision.
It includes also the calculation of the Standard Corrective Sigma according to the pages 43 and 50.
Sigma is a Pythagorean Mean and means also an overall measure of all measurement deviations.
And by means of a comparison of the 8 individual relative correctives with one another, any possible incorrect reading of a measurement value can be discovered.

# Table of Contents

Pages

1 **Title**

2 **Copyright**

3 **Acknowledgement, Dedication**

4 **Abstractum**

6 **A biographical Foreword, Remarks concerning the english Text**

7 **List of Names**

8 **Table of Contents**

11 **Introduction:** Functionality of Electromagnetic Transformers
12 About Complex Amplitudes, Vectors and Transformers
13 The rediscovered Title and Author of a stimulating Book of my Father
14 The Mutual Complex Interaction
15 A holiday accident in the Village Smithy, and Memories of My School Days
16 About Logic, Dialectics and Mathematical Exactness
17 End of Second World War and Childhood
18 Immigration to West Germany, three work accidents
19 Study with the Help from the Social Office
20 Painful Course in Ffm, but fruitful Development in Finland
21 Mathematical Achievements after My Retirement

22 **Chapter 1:** The **Coherent Determination** of the Mutual Inductive and the Mutual Proximity-Effect Coupling Coefficient

22 1.1 All calculations are based on the two complex equations of the two-port network theory including the real part $R_m$ of the complex coefficients $B = C$

23 1.2 The proof of the equality of the complex coefficients $B = C$

24 1.3 The definition of the complex coupling coefficient $k$

26 1.4 The coherent determination of both the mutual inductive coupling coefficient $k_{ind}$ and the mutual Proximity-Effect coupling coefficient $k_{prox}$

28 1.5 The separated coupling coefficients $k_{ind}$ and $k_{prox}$ as functions of the complex coupling coefficient $k$ and the quality factors $Q_1$ and $Q_2$

29 1.6 The real component $k_{real}$ and the imaginary component $k_{im}$ of the complex coupling coefficient $k$ as functions of the separated coupling coefficients $k_{ind}$ and $k_{prox}$ and the quality factors $Q_1$ and $Q_2$

31 1.7 The complex-geometrical mean $G$

34 1.8 General remarks on the convertion from rectangular Cartesian coordinates $x$ and $y$ to polar coordinates $r$ and $\theta$

35 **Chapter 2:** The **C**omplex **R**elative-**E**rror **E**qualization **T**heory, **CREET**

45 **Chapter 3:** Measurement Values of an Experimentation Transformer

46 **Chapter 4:** Application of the **C**omplex **R**elative-**E**rror **E**qualization **T**heory to the Measurement Values of the Experimentation Transformer

46 4.1 The complex-geometrical mean of both impedance ratios (primary coil and secondary coil) of the experimentation transformer of Chapter 3
48 4.1p Equalization of the impedance measurement values of the primary side
49 4.1s Equalization of the impedance measurement values of the secondary side
50 4.2 Comparison of the original with the equalized measurement values
50 4.3 The Standard Corrective Sigma = $\sigma$ of the CREET
51 4.4 Calculations of $k_g$, $B_e$, $R_{me}$, $M_e$, $k_{proxe}$, $k_{inde}$

# Table of Contents

54 **Chapter 5:** Coherent Determination of the Mutual Inductance M and the Mutual Resistance $R_m$ by Means of the Complex Coupling Coefficient k and the Complex-Geometrical Mean of A and D

56 **Chapter 6:** The One-Path Formulae for the Homogeneous Use of the Two Open-Circuit and the Two Short-Circuit Impedance Measurement Values for the Coherent Determination of the Mutual Inductance and Resistance

56 6.1 Complex derivation of non-complex formulae for $M$ and $R_m$

58 6.2 Application to unequalized impedance measurement values

59 6.3 Application to equalized impedance measurement values

60 **Chapter 7:** The Two-Path Formulae for Parallel Coherent Calculation of Both the Mutual Inductance and the Mutual Resistance

60 7.1 Determination of $M_u$ and $R_{mu}$ from unequalized measurement values

62 7.2 Determination of $M_e$ and $R_{me}$ from equalized measurement values

63 **Chapter 8:** Coherent Derivation of the Complex Transformation Ratio and the Complex Coupling Coefficient by Means of Measurements of two Complex Voltage Ratios

64 **Chapter 9:** The Complex Conversion from Any Transformer to a Symmetrical Model for the Complex Derivation of the Two Leakage Impedances

66 **Chapter 10:** The Derivation of the Universal Function for the Determination of the Maximum Possible Power Transfer Efficiency $\eta_{max}$

66 10.1 The physical fundament for the derivation of the $\eta_{max}$-formula

66 10.2 The derivation of the maximum efficiency $\eta_{max}$

72 10.3 An unsuccessful search for an alternative $\eta_{max}$-formula

74 10.4 An example for the calculation of $C_{opt}$, $R_{opt}$ and $\eta_{max}$

75 10.5 A verification of the analytically derived $\eta_{max}$-formula

77 **Chapter 11:** The Experimental Proof of the Mutual Proximity-Effect from 1 to 200 kHz With and Without the Ferrite Core ETD 34

77 11.1 A simulation of the mutual Proximity-Effect by two parallel resistors

81 11.2 The construction of the Proximity-Effect experimentation transformer

82 11.3 Measurements of the Proximity-Effect experimentation transformer

82 11.4 Calculation results by means of the BASIC-program of Chapter 15

84 11.5 Frequency dependence of the muutual resistance $R_m$ [Ω]

85 11.6 Frequency dependence of the Proximity-Effect coupling coefficient $k_{prox}$

86 11.7 Frequency dependence of the inductive coupling coefficient $k_{ind}$

87 **Chapter 12:** An Hypothetical Assumption of a Very Large Mutual Proximity-Effect for the Purpose of Studying only

# Table of Contents

89 **Chapter 13:** An Application of the $C_{opt}$, $R_{opt}$ and $\eta_{max}$-Transformer Formulae to Two Complex Two-Port $\Pi$-Networks with Discrete Components

89 13.1 A two-port $\Pi$-network consisting of one "vertical" inductance $L_a$ (input side) , one "horizontal" resistance $R_h$ and a parallel connection (output) of "vertical" $R_b$ with $L_b$

93 13.2a A two-port $\Pi$-network consisting of one "horizontal" inductance $L$ and two "vertical" resistances $R_a$ and $R_b$

97 13.2b Comparison of the results of the formulae for $C_{opt}$, $R_{opt}$ and $\eta_{max}$ , derived in Chapter 10, with the foregoing classical calculation of the Chapter 13.2a

101 **Chapter 14:** The General Two-Path Method with Roughly Evaluated Terminating Resistances for the Coherent Determination of Both the Mutual Inductance and the Mutual Resistance

(Because the derived formulae are non-complex, common scientific calculators can be used, even also a slide-rule on principle.)

101 14.1 The two alternative cases that the secondary coil and the primary coil are one after another terminated with roughly chosen resistances

105 14.2 A numerical verification of the derived non-complex formulae

107 14.3 Extrapolation of the input impedance at short-circuited output, & vice versa

110 **Chapter 15:** A BASIC-Program for the SHARP Pocket Computer PC-1403H, with CREET and Coherent Utilization of Transformer Measurements

110 15.1 The calculation program with some additional explanations: Equalization of the Transformer Impedance Measurement Values and Calculation of the Quality Factors of the Primary- and the Secondary Coil and the Coherent Determination of Both the Mutual Inductance and the Mutual Proximity-Effect Resistance and furthermore Calculation of the Corresponding Coupling Coefficients and the Determination of the Optimum Terminating Parallel Capacitance and the optimum Parallel Load Resistance for **Maximum Power Transfer Efficiency**

119 15.2 The BASIC-program to print the calculation results

121 15.3 Results with the PC-1403H for the transformer of Chapter 3

122 15.4 Results with the PC-1403H for the two-port $\Pi$-network of Chapter 13.1

123 15.5 Results with the PC-1403H for the two-port $\Pi$-network of Chapter 13.2b

124 **Appendix 1:** A Numerical Verification of the CREET that the Sum of the Primary Side Related 4 Relative Correctives is 0ppositely Equal to the Sum of the Squares of These

125 **Appendix 2:** A Numerical Verification of the CREET that the Sum of the Secondary Side Related 4 Relative Correctives is 0ppositely Equal to the Sum of the Squares of These

126 to 131 **Appendix 3:** An Algebraical Proof of the Characteristic of the CREET, that the Arithmetical Mean is Always Oppositely Equal to the Square of the Standard Corrective of the 4 Relative Correctives of Each Side of a Transformer

# Chapter 1

# The Coherent Determination of the Mutual Inductive and the Mutual Proximity-Effect Coupling Coefficient

1.1 All calculations are based on the two complex equations of the two-port network theory including the real part $R_m$ of the complex coefficients $B = C$ :

$$U_1 = (R_1 + i\cdot\omega\cdot L_1)\cdot I_1 + (R_m + i\cdot\omega\cdot M)\cdot I_2 = A\cdot I_1 + B\cdot I_2$$

$$U_2 = (R_m + i\cdot\omega\cdot M)\cdot I_1 + (R_2 + i\cdot\omega\cdot L_2)\cdot I_2 = C\cdot I_1 + D\cdot I_2$$

On the right hand side are given the general equations of the two-port network theory, called also the four-terminal network theory, in German *"Vierpoltheorie"*, that is applied here to electromagnetic transformers: All four coefficients $A$ , $B$ , $C$ and $D$ can be generally complex.
The direction of the two currents $I_1$ [A] and $I_2$ [A] is positive by definition, if they flow towards the transformer. This means: If both currents are increasing, they induce a positive voltage $U_1$ [V] of the primary coil and a positive voltage $U_2$ [V] of the secondary coil. $\omega = 2\cdot\pi\cdot f$ is the angular frequency, during $f$ [Hz] is the frequency and $T$ [sec] is the period time according to $f\cdot T = 1$ .
All voltages and currents are assumed to be sine-shaped, but more or less phase-shifted.
The complex coefficients $B$ and $C$ of the two-port equations are equal to each other and identical with the mutual impedance $(R_m + i\cdot\omega\cdot M)$ . $R_m$ is the real part and $i\cdot\omega\cdot M$ is the imaginary part.

| | | |
|---|---|---|
| $R_1$ = open-circuit resistance [Ω] , | $L_1$ = open-circuit inductance [H] | of the primary coil |
| $R_2$ = open-circuit resistance [Ω] , | $L_2$ = open-circuit inductance [H] | of the secondary coil |
| $R_m$ = mutual resistance [Ω] , | $M$ = mutual inductance [H] | of mutual interaction |

During the measurement of the open-circuit impedance $Z_1 = R_1 + i\cdot\omega\cdot L_1$ , the secondary coil is unloaded and the current $I_2 = 0$ . Therefore the coefficient $A = Z_1$ .
During the measurement of the open-circuit impedance $Z_2 = R_2 + i\cdot\omega\cdot L_2$ , the primary coil is unloaded and the current $I_1 = 0$ . Therefore the coefficient $D = Z_2$ .

For the determination of the coefficients $B$ and $C$ are needed additionally the measurements of the short-circuit impedances $Z_{1sc}$ and $Z_{2sc}$ . For the short-circuited secondary coil is $U_2 = 0$ , and it follows from the second two-port equation:

$$I_2 = \frac{-C}{D}\cdot I_1 \quad ,$$

and insertion into the first two-port equation yields:

$$\frac{B\cdot C}{D} = A - Z_{1sc} \qquad B\cdot C = D\cdot(A - Z_{1sc}) = D\cdot A\cdot\left(1 - \frac{Z_{1sc}}{A}\right) = A\cdot D\cdot\left(1 - \frac{Z_{1sc}}{Z_1}\right)$$

Analogously is valid also:

$$\frac{C\cdot B}{A} = D - Z_{2sc} \qquad C\cdot B = A\cdot(D - Z_{2sc}) = A\cdot D\cdot\left(1 - \frac{Z_{2sc}}{D}\right) = A\cdot D\cdot\left(1 - \frac{Z_{2sc}}{Z_2}\right)$$

Because of the equality of $B\cdot C = C\cdot B$ , it follows exactly the proportionality equation:

$$\frac{Z_{1sc}}{Z_1} = \frac{Z_{2sc}}{Z_2}$$

## 1.4 The Coherent determination of both the mutual inductive coupling coefficient $k_{ind}$ and the mutual Proximity-Effect coupling coefficient $k_{prox}$ :

As a conclusion from the first two pages of this Chapter 1, it can be written:

$$B = \pm\sqrt{Z_1 \cdot Z_2 \cdot \left(1 - \frac{Z_{1sc}}{Z_1}\right)} \qquad \text{respectively} \qquad C = \pm\sqrt{Z_1 \cdot Z_2 \cdot \left(1 - \frac{Z_{2sc}}{Z_2}\right)}$$

And again arises the question concerning the sign of the square root. But now it will be solved by means of polar coordinates. In order to decompose $B$ , the square root has to be squared at first:

$$B^2 = \left(R_2 + i\cdot\omega\cdot L_2\right)\cdot\left[\left(R_1 + i\cdot\omega\cdot L_1\right) - \left(R_{1sc} + i\cdot\omega\cdot L_{1sc}\right)\right]$$

$$B^2 = \left(R_2 + i\cdot\omega\cdot L_2\right)\cdot\left[\left(R_1 - R_{1sc}\right) + i\cdot\omega\cdot\left(L_1 - L_{1sc}\right)\right]$$

$$B^2 = \left[R_2\cdot\left(R_1 - R_{1sc}\right) - \omega\cdot L_2\cdot\omega\cdot\left(L_1 - L_{1sc}\right)\right] + i\cdot\left[\omega\cdot L_2\cdot\left(R_1 - R_{1sc}\right) + R_2\cdot\omega\cdot\left(L_1 - L_{1sc}\right)\right]$$

$$a = R_2\cdot\left(R_1 - R_{1sc}\right) - \omega\cdot L_2\cdot\omega\cdot\left(L_1 - L_{1sc}\right) \qquad b = \omega\cdot L_2\cdot\left(R_1 - R_{1sc}\right) + R_2\cdot\omega\cdot\left(L_1 - L_{1sc}\right)$$

$$\left|B^2\right| = \sqrt{a^2 + b^2} \qquad \beta = \operatorname{atan}\left(\frac{b}{a}\right) + (a < 0)\cdot(b > 0)\cdot\pi + (a < 0)\cdot(b < 0)\cdot\pi + (a > 0)\cdot(b < 0)\cdot 2\cdot\pi$$

If for instance $a$ would be positive, but $b$ negative, then the complex vector $B^2$ is situated in the fourth quadrant, and $2\cdot\pi$ is added to the arcus tangens atan.
The square root of a power function is extracted by dividing its exponent by $2$ .

$$B^2 = \left|B^2\right|\cdot e^{i\cdot\beta} \qquad \phi = \frac{\beta}{2} \qquad B = B_r + i\cdot B_i = \sqrt{\left|B^2\right|}\cdot e^{i\cdot\phi} = \sqrt{\left|B^2\right|}\cdot\left(\cos(\phi) + i\cdot\sin(\phi)\right)$$

$$B_r = \sqrt{\left|B^2\right|}\cdot\cos(\phi) = \sqrt[4]{a^2 + b^2}\cdot\cos(\phi) \qquad B_i = \sqrt{\left|B^2\right|}\cdot\sin(\phi) = \sqrt[4]{a^2 + b^2}\cdot\sin(\phi)$$

Since the calculation of $B$ is based on measurements of the impedance of the primary coil at short-circuited secondary coil, the subscript 1 may be used for a more precise identification of the derived $R_m$ and $M$ , and furthermore also for $k_{prox}$ and $k_{ind}$ as well:

$$B = R_{m1} + i\cdot\omega\cdot M_1 \qquad R_{m1} = \operatorname{Re}(B) = B_r \qquad \omega\cdot M_1 = \operatorname{Im}(B) = B_i$$

$$R_{m1} = \sqrt[4]{a^2 + b^2}\cdot\cos(\phi) \qquad M_1 = \frac{1}{\omega}\cdot\sqrt[4]{a^2 + b^2}\cdot\sin(\phi)$$

$$k_{prox1} = \frac{R_{m1}}{\sqrt{R_1\cdot R_2}} \qquad k_{ind1} = \frac{M_1}{\sqrt{L_1\cdot L_2}}$$

The sign of the values of $R_{m1}$ and $M_1$ depends only upon the angle $\phi$ of the trigonometric functions $\cos(\phi)$ and $\sin(\phi)$ respectively. The fourth power root is related to the square $\left(a^2 + b^2\right)$ of the magnitude of the vector $B$ .
The same kind of calculation can be carried out with respect to measurements of the impedance of the secondary coil at short-circuited primary coil. Only the subscripts 1 and 2 have to be interchanged, and $B$ , $\phi$ , $a$ and $b$ have to be replaced by other letters, for instance $C$ , $\psi$ , $g$ and $h$ .

$$C^2 = (R_1 + i\cdot\omega\cdot L_1)\cdot\left[(R_2 + i\cdot\omega\cdot L_2) - (R_{2sc} + i\cdot\omega\cdot L_{2sc})\right]$$

$$C^2 = (R_1 + i\cdot\omega\cdot L_1)\cdot\left[(R_2 - R_{2sc}) + i\cdot\omega\cdot(L_2 - L_{2sc})\right]$$

$$C^2 = \left[R_1\cdot(R_2 - R_{2sc}) - \omega\cdot L_1\cdot\omega\cdot(L_2 - L_{2sc})\right] + i\cdot\left[\omega\cdot L_1\cdot(R_2 - R_{2sc}) + R_1\cdot\omega\cdot(L_2 - L_{2sc})\right]$$

$$g = R_1\cdot(R_2 - R_{2sc}) - \omega\cdot L_1\cdot\omega\cdot(L_2 - L_{2sc}) \qquad h = \omega\cdot L_1\cdot(R_2 - R_{2sc}) + R_1\cdot\omega\cdot(L_2 - L_{2sc})$$

$$\left|C^2\right| = \sqrt{g^2 + h^2} \qquad \gamma = \operatorname{atan}\left(\frac{h}{g}\right) + (g < 0)\cdot(h > 0)\cdot\pi + (g < 0)\cdot(h < 0)\cdot\pi + (g > 0)\cdot(h < 0)\cdot 2\cdot\pi$$

If for instance $g$ would be positive, but $h$ negative, then the complex vector $C^2$ is situated in the fourth quadrant, and $2\cdot\pi$ is added to the arcus tangens atan.
The square root of a power function is extracted by dividing its exponent by 2 .

$$C^2 = \left|C^2\right|\cdot e^{i\cdot\gamma} \qquad \psi = \frac{\gamma}{2} \qquad C = C_r + i\cdot C_i = \sqrt{\left|C^2\right|}\cdot e^{i\cdot\psi} = \sqrt{\left|C^2\right|}\cdot(\cos(\psi) + i\cdot\sin(\psi))$$

$$C_r = \sqrt{\left|C^2\right|}\cdot\cos(\psi) = \sqrt[4]{g^2 + h^2}\cdot\cos(\psi) \qquad C_i = \sqrt{\left|C^2\right|}\cdot\sin(\psi) = \sqrt[4]{g^2 + h^2}\cdot\sin(\psi)$$

Since the calculation of $C$ is based on measurements of the impedance of the secondary coil at short-circuited primary coil, the subscript 2 may be used for a more precise identification of the derived $R_m$ and $M$, and furthermore also for $k_{prox}$ and $k_{ind}$ as well:

$$C = R_{m2} + i\cdot\omega\cdot M_2 \qquad R_{m2} = \operatorname{Re}(C) = C_r \qquad \omega\cdot M_2 = \operatorname{Im}(C) = C_i$$

$$R_{m2} = \sqrt[4]{g^2 + h^2}\cdot\cos(\psi) \qquad M_2 = \frac{1}{\omega}\cdot\sqrt[4]{g^2 + h^2}\cdot\sin(\psi)$$

$$k_{prox2} = \frac{R_{m2}}{\sqrt{R_1\cdot R_2}} \qquad k_{ind2} = \frac{M_2}{\sqrt{L_1\cdot L_2}}$$

The sign of the values of $R_{m2}$ and $M_2$ depends only upon the angle $\psi$ of the trigonometric functions $\cos(\psi)$ and $\sin(\psi)$ respectively. The fourth power root is related to the square $(g^2 + h^2)$ of the magnitude of the vector $C$. Theoretically the corresponding results of the two calculations should be equal to each other, because of $C = B$. But accidental measurement errors as small deviations from the true values of the short-circuit- and the open-circuit impedance values are always unavoidable.

In other Chapters of this work are described also other methods without the means of polar coordinates. Such methods can be used, if the correct sign of the values of the mutual resistance $R_m$ and the mutual inductance $M$ are clear already a priori , as it is the normal case in the application of electromagnetic transformers, enough below of their own resonance frequencies. But if other two-port networks are considered, then polar coordinates have to be applied in such a way as just shown here.

Finally a physical hint could be given still that really $R_m$, and not $M$ alone, is an existing parameter of the mutual property of an electromagnetic transformer:
If namely for the impedance measurements an external inductor and / or an external resistor would be connected in series with the primary- or the secondary coil - or two inductors and / or two resistors, each of both in series with one of both transformer coils - in every case the mutual impedance would stay absolutely uneffected. But, of course, the overall coupling coefficient is decreased then.

# Chapter 3

## Measurement Values of an Experimentation Transformer

The following complex measurement values are small signal values and are measured at one self-wound and not necessarily small transformer, provided with the Siemens ferrite core of the material N 41 and the size EE 21-9-5 with internal air gap of 120 µm, AL = 200 nH per turn square.
The coilformer is provided with two equal sections, the one is for the primary coil, the other for the secondary coil. Between both coils is a partition wall of measured 0.91 mm. The width of each section is 4.27 mm. The winding bottom has a measured square of 7.48 mm x 7.48 mm.
The primary coil consists of 400 turns of 0.16 EJF2 double enamelled copper wire and the the secondary coil consists of 70 turns of 0.40 EJF2 double enamelled copper wire.
Concerning the published AL-value the inductances should be at least approximately:

$N_1 := 400 \qquad L_1 = 200 \cdot nH \cdot 400^2 = 32 \cdot mH \qquad N_2 := 70 \qquad L_2 = 200 \cdot nH \cdot 70^2 = 980 \cdot \mu H$

The actual values can be obtained only be means of measurements with an impedance analyser:
The technical term "open-circuit" means that during the measurement of the one of both coils, the at this moment other coil is not connected to any load. Therefore in this case no additional subscript is used. But the technical term "short-circuit" [subscript sc] means that during the measurement of the one of both coils, the at this moment other coil is short-circuited.

Used measurement frequency: $f := 60 \cdot kHz \qquad \omega := 2 \cdot \pi \cdot f \qquad \omega = 3.769911 \times 10^5 \, s^{-1}$

Measured no-load and short-circuited resistances and inductances of the primary coil:

$R_1 := 45 \cdot \Omega \qquad R_{1sc} := 72 \cdot \Omega$

$L_1 := 32.5 \cdot mH \qquad L_{1sc} := 3.84 \cdot mH$

Measured no-load and short-circuited resistances and inductances of the secondary coil:

$R_2 := 2.5 \cdot \Omega \qquad R_{2sc} := 2.34 \cdot \Omega$

$L_2 := 1006 \cdot \mu H \qquad L_{2sc} := 118 \cdot \mu H$

The index "sc" stands for short-circuited.

## Application of the Formulae derived in Chapter 1.4 to the measured Values of an Experimentation Transformer of the previous Chapter 3, Calculation with Polar Coordinates:

<u>First Path</u>: The primary coil is measured.

$$a := R_2 \cdot (R_1 - R_{1sc}) - \omega \cdot L_2 \cdot \omega \cdot (L_1 - L_{1sc}) \qquad b := \omega \cdot L_2 \cdot (R_1 - R_{1sc}) + R_2 \cdot \omega \cdot (L_1 - L_{1sc})$$

$$a = -4.097732 \times 10^6 \, \Omega \cdot \Omega \qquad b = 1.677158 \times 10^4 \, \Omega \cdot \Omega$$

$$\beta := \operatorname{atan}\left(\frac{b}{a}\right) + (a < 0) \cdot (b > 0) \cdot \pi + (a < 0) \cdot (b < 0) \cdot \pi + (a > 0) \cdot (b < 0) \cdot 2 \cdot \pi$$

$$\beta = 3.137500 \, rad \qquad \beta = 179.765496 \, Grad$$

$$\phi := \frac{\beta}{2} \qquad B_r := \sqrt[4]{a^2 + b^2} \cdot \cos(\phi) \qquad B_i := \sqrt[4]{a^2 + b^2} \cdot \sin(\phi)$$

$$R_{m1} := B_r \qquad R_{m1} = 4.142584 \, \Omega \qquad M_1 := \frac{B_i}{\omega} \qquad M_1 = 5.369595 \, mH$$

$$k_{prox1} := \frac{R_{m1}}{\sqrt{R_1 \cdot R_2}} \qquad k_{prox1} = 0.390567 \qquad k_{ind1} := \frac{M_1}{\sqrt{L_1 \cdot L_2}} \qquad k_{ind1} = 0.939076$$

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Second Path: The secondary coil is measured.

$g := R_1 \cdot (R_2 - R_{2sc}) - \omega \cdot L_1 \cdot \omega \cdot (L_2 - L_{2sc})$ $\qquad$ $h := \omega \cdot L_1 \cdot (R_2 - R_{2sc}) + R_1 \cdot \omega \cdot (L_2 - L_{2sc})$

$g = -4.101642 \times 10^{6}\, \Omega \cdot \Omega$ $\qquad$ $h = 1.702492 \times 10^{4}\, \Omega \cdot \Omega$

$$\gamma := \operatorname{atan}\left(\frac{h}{g}\right) + (g < 0) \cdot (h > 0) \cdot \pi + (g < 0) \cdot (h < 0) \cdot \pi + (g > 0) \cdot (h < 0) \cdot 2 \cdot \pi$$

$\gamma = 3.137442\,\text{rad}$ $\qquad$ $\gamma = 179.762181\,\text{Grad}$

$\psi := \frac{\gamma}{2}$ $\qquad$ $C_r := \sqrt[4]{g^2 + h^2} \cdot \cos(\psi)$ $\qquad$ $C_i := \sqrt[4]{g^2 + h^2} \cdot \sin(\psi)$

$R_{m2} := C_r$ $\qquad$ $R_{m2} = 4.203153\,\Omega$ $\qquad$ $M_2 := \frac{C_i}{\omega}$ $\qquad$ $M_2 = 5.372157\,\text{mH}$

$k_{prox2} := \frac{R_{m2}}{\sqrt{R_1 \cdot R_2}}$ $\qquad$ $k_{prox2} = 0.396277$ $\qquad$ $k_{ind2} := \frac{M_2}{\sqrt{L_1 \cdot L_2}}$ $\qquad$ $k_{ind2} = 0.939524$

# Chapter 7

## The *Two-Path Formulae* for Parallel Coherent Calculation of both the Mutual Inductance and the Mutual Resistance

### 7.1 Determination of $M_u$ and $R_{mu}$ from *unequalized* measurement values:

The following formulae are derived from the four-terminal equations of Chapter 1 with respect to the open-circuit and short-circuit measurements and the application of the solution of the mixed quadratic equation $x^2 + p \cdot x + q = 0$ . The sequence of the individual formulae given here is developed for the straightforward calculation process. If instead of short-circuit measurements the output of a transformer is loaded with certain external resistances $R_{ex1}$ and $R_{ex2}$ respectively, please see Chapter 14.

$Q_1 := \frac{\omega \cdot L_1}{R_1}$ $\qquad$ $Q_1 = 272.271363$ $\qquad$ $Q_2 := \frac{\omega \cdot L_2}{R_2}$ $\qquad$ $Q_2 = 151.701226$

$S_1 := R_1^2 + \omega^2 \cdot L_1^2$ $\qquad$ $H_1 := Q_1 + \frac{1}{Q_1}$ $\qquad$ $S_2 := R_2^2 + \omega^2 \cdot L_2^2$ $\qquad$ $H_2 := Q_2 + \frac{1}{Q_2}$

First path:

$\xi_1 := (R_1 - R_{1sc}) \cdot S_2$

$\eta_1 := \omega \cdot (L_1 - L_{1sc}) \cdot S_2$

$$\Lambda_1 := \frac{\eta_1}{R_2} - \frac{\xi_1}{\omega \cdot L_2}$$

$$\Pi_1 := \sqrt{\frac{1}{R_2^2} + \frac{1}{\omega^2 \cdot L_2^2}} \cdot \sqrt{\xi_1^2 + \eta_1^2}$$

$$\Gamma_1 := \frac{\Lambda_1 + \Pi_1}{2}$$

Second path:

$\xi_2 := (R_2 - R_{2sc}) \cdot S_1$

$\eta_2 := \omega \cdot (L_2 - L_{2sc}) \cdot S_1$

$$\Lambda_2 := \frac{\eta_2}{R_1} - \frac{\xi_2}{\omega \cdot L_1}$$

$$\Pi_2 := \sqrt{\frac{1}{R_1^2} + \frac{1}{\omega^2 \cdot L_1^2}} \cdot \sqrt{\xi_2^2 + \eta_2^2}$$

$$\Gamma_2 := \frac{\Lambda_2 + \Pi_2}{2}$$

$$M_1 := \frac{1}{\omega}\cdot\sqrt{\frac{\Gamma_1}{H_2}} \qquad R_{m1} := \frac{\frac{\xi_1}{R_2} + \frac{\eta_1}{\omega\cdot L_2}}{2\cdot H_2\cdot\omega\cdot M_1} \qquad M_2 := \frac{1}{\omega}\cdot\sqrt{\frac{\Gamma_2}{H_1}} \qquad R_{m2} := \frac{\frac{\xi_2}{R_1} + \frac{\eta_2}{\omega\cdot L_1}}{2\cdot H_1\cdot\omega\cdot M_2}$$

$$M_1 = 5.369595\,\text{mH} \qquad R_{m1} = 4.142584\,\Omega \qquad M_2 = 5.372157\,\text{mH} \qquad R_{m2} = 4.203153\,\Omega$$

As a first proof, the values of both pairs of the results are rather equal to each other:

$$2\cdot\left(\frac{M_1 - M_2}{M_1 + M_2}\right) = -4.769752\times 10^{-4} \qquad 2\cdot\left(\frac{R_{m1} - R_{m2}}{R_{m1} + R_{m2}}\right) = -0.014515$$

However, the relative difference between both $R_m$-results is here about 30 times larger than the relative difference of the $M$-results. The cause is surely that the resistances of the measured short-circuit and open-circuit impedances are much smaller than the reactances and therefore relatively less accurate.
But if at the beginning the fine **C**omplex **R**elative**-E**rror **E**qualization **T**heory, the **CREET** of Chapter 2, would have been applied, then the **differences between Path One and Path Two would be zero.**
A detailed application of the CREET is given in my earlier report of 06 March 2022 about
"Maximization of the Efficiency of a Transformer, solely by Optimizing its Load Impedance"
But here in the following, the calculation is continued with the arithmetical mean values:

$$R_m := \frac{R_{m1} + R_{m2}}{2} \qquad R_m = 4.172869\,\Omega \qquad M := \frac{M_1 + M_2}{2} \qquad M = 5.370876\,\text{mH}$$

$$(R_{me} = 4.179597\cdot\Omega) \qquad (M_e = 5.370868\cdot\text{mH})$$

$$k_{prox} := \frac{R_m}{\sqrt{R_1\cdot R_2}} \qquad k_{prox} = 0.393422 \qquad k_{ind} := \frac{M}{\sqrt{L_1\cdot L_2}} \qquad k_{ind} = 0.939300$$

$$(k_{proxe} = 0.394076) \qquad (k_{inde} = 0.939301)$$

In order to demonstrate the result of my calculations, the following three formulae are copied from Chapter 10.4 , page 74 , of my book from 2014, but here calculated from unequalized measurement values, it means without the CREET.

The optimum terminating capacitance is:

$$C_{opt} := \frac{1 - k_{ind}\cdot k_{prox}\cdot\sqrt{\frac{Q_1}{Q_2}}}{\left(k_{ind}\cdot\sqrt{\frac{Q_1}{Q_2}} - k_{prox}\right)^2 + \left(1 - k_{prox}^2\right)\cdot\left(1 + \frac{1}{Q_2^2}\right)}\cdot\frac{1}{\omega^2\cdot L_2}$$

$$C_{opt} = 2.216370\,\text{nF}$$
===============

$$(C_{opte} = 2.208199\,\text{nF})$$

For a comparison, the results written in brackets and additionally market with the subscript "e" originate from my book mentioned on the title page of this 12 pages. The subscript "e" stands for equalized.

The optimum terminating resistance is:

$$R_{opt} := \frac{\left(k_{ind}\cdot\sqrt{\frac{Q_1}{Q_2}} - k_{prox}\right)^2 + \left(1 - k_{prox}^2\right)\cdot\left(1 + \frac{1}{Q_2^2}\right)}{\sqrt{\left(1 + k_{ind}^2\cdot Q_1\cdot Q_2\right)\cdot\left(1 - k_{prox}^2\right)}}\cdot Q_2\cdot\omega\cdot L_2$$

$$R_{opt} = 0.522340\,\text{k}\Omega$$
================

$$(R_{opte} = 0.521067\cdot\text{k}\Omega)$$

The maximum possible power transfer efficiency is:

$$\eta_{max} := \frac{k_{ind}^2\cdot Q_1\cdot Q_2 + k_{prox}^2}{\left(\sqrt{1 + k_{ind}^2\cdot Q_1\cdot Q_2} + \sqrt{1 - k_{prox}^2}\right)^2}$$

$$\eta_{max} = 0.990414$$
=============

$$(\eta_{maxe} = 0.990418)$$