Maximization of the Efficiency of a Transformer, solely by Optimizing its Load Impedance, which consists of a Resistor and a Parallel Capacitor

This task is solved in all mathematical details in my book:

"Coherent Mathematical Treatment
of the Inductive and Resistive Coupling
Acting in Electromagnetic Transformers"

But in the short version of my book compiled here, only the calculation of one example is shown, i. e. without the derivation of all formulas used for it as well as without the many parallel mathematical secondary investigations.

My book was printed and published under ISBN 978-952-93-4410-9 in 2014 by Unigrafia in Helsinki. Unfortunately, it is no longer available due to lack of general interest in mathematical details of an electromagnetic transformer. Only in some university libraries it found a polite reception.
My book comprises 131 pages in A4 format and is intended only for readers who enjoy the complex four-pole theory and the associated mathematically based measurement methods.

But nevertheless, at first a preface for the understanding of the task: With the above named title of my book I mean especially the fact, according my opinion, that beside of the well-known mutual inductance also a mutual resistance exists, which can be deduced exactly from the measured open-loop and short circuit impedances. The physical reason of the mutual resistance is the mutual Proximity-Effect, which increases with increasing frequency of the alternating current, but, of course, is zero at direct current.

Differently from general use, an "ideal" transformer means in my book not a losfree transformer, but a transformer that can be described by inductances and resistances, i. e. without internal capacitances, according to both four-pole equations. Such a transformer has the property that the ratio of the short circuit impedance to the open-loop impedance, measured at the primary coil, is equal to the corresponding ratio at the secondary coil.

But practical experiences show that both measured ratios are not always exactly equal to each other, because of natural measurement deviations, also due to internal capacitances. For the obtainment of the best possible approximation to an "ideal" transformer, as defined above, the idea described in my book is to form the complex geometrical average value of these two ratios, in order to convert then all individually measured impedances by means of a minimization of the sum of the squares of all relative deviations, as if the converted impedances originated from a transformer, which perfectly corresponds to the two complex four-pole equations that contain per definition only inductances and resistances.

Additionally and independently of this correction of all individually measured impedances, I succeeded by definition of a mutual resistive coupling factor, as shown below, a deduction of a clearly structured elementary formula for the calculation of the maximum possible efficiency of a transformer. This formula also given below contains only the following 4 parameters, namely: The quality factors Q_1 and Q_2, the inductive coupling factor k_{ind} and perhaps as a novelty the resistive coupling factor k_r.

$$Q_1 = \frac{\omega \cdot L_1}{R_1} \qquad Q_2 = \frac{\omega \cdot L_2}{R_2} \qquad k_{ind} = \frac{M}{\sqrt{L_1 \cdot L_2}} \qquad k_r = \frac{R_m}{\sqrt{R_1 \cdot R_2}}$$

$$\eta_{max} = \frac{k_{ind}^2 \cdot Q_1 \cdot Q_2 + k_r^2}{\left(\sqrt{1 + k_{ind}^2 \cdot Q_1 \cdot Q_2} + \sqrt{1 - k_r^2}\right)^2}$$

The properties of this formula are very clear: The efficiency η_{max} never can be larger than 1.

The resistive coupling factor k_r increases always somewhat the efficiency.

And with the assumption of $k_{ind} = 1$ and $k_r = 0$ a fast estimation of the efficiency is possible.

Chapter 3

Measurement Values of an Experimentation Transformer

The following complex measurement values are small signal values and are measured at one self-wound and not necessarily small transformer, provided with the Siemens ferrite core of the material N 41 and the size EE 21-9-5 with internal air gap of 120 µm, AL = 200 nH per turn square.
The coilformer is provided with two equal sections, the one is for the primary coil, the other for the secondary coil. Between both coils is a partition wall of measured 0.91 mm. The width of each section is 4.27 mm. The winding bottom has a measured square of 7.48 mm x 7.48 mm.
The primary coil consists of 400 turns of 0.16 EJF2 double enamelled copper wire and the the secondary coil consists of 70 turns of 0.40 EJF2 double enamelled copper wire.
Concerning the published AL-value the inductances should be at least approximately:

$N_1 := 400 \qquad L_1 = 200 \cdot \mathrm{nH} \cdot 400^2 = 32 \cdot \mathrm{mH} \qquad N_2 := 70 \qquad L_2 = 200 \cdot \mathrm{nH} \cdot 70^2 = 980 \cdot \mu\mathrm{H}$

The actual values can be obtained only be means of measurements with an impedance analyser:
The technical term "open-circuit" means that during the measurement of the one of both coils, the at this moment other coil is not connected to any load. Therefore in this case no additional subscript is used. But the technical term "short-circuit" [subscript sc] means that during the measurement of the one of both coils, the at this moment other coil is short-circuited.

Used measurement frequency: $f := 60 \cdot \mathrm{kHz} \qquad \omega := 2 \cdot \pi \cdot f \qquad \omega = 3.769911 \times 10^5\,\mathrm{Hz}$

Measured resistances and inductances of the primary coil:

$R_1 := 45 \cdot \Omega \qquad a_1 := R_1 \qquad a_1 = 45\,\Omega$

$L_1 := 32.5 \cdot \mathrm{mH} \qquad b_1 := \omega \cdot L_1 \qquad b_1 = 1.225221 \times 10^4\,\Omega$

$R_{1sc} := 72 \cdot \Omega \qquad a_3 := R_{1sc} \qquad a_3 = 72\,\Omega$

$L_{1sc} := 3.84 \cdot \mathrm{mH} \qquad b_3 := \omega \cdot L_{1sc} \qquad b_3 = 1.447646 \times 10^3\,\Omega$

Measured resistances and inductances of the secondary coil:

$R_2 := 2.5 \cdot \Omega \qquad a_2 := R_2 \qquad a_2 = 2.5\,\Omega$

$L_2 := 1006 \cdot \mu\mathrm{H} \qquad b_2 := \omega \cdot L_2 \qquad b_2 = 3.792531 \times 10^2\,\Omega$

$R_{2sc} := 2.34 \cdot \Omega \qquad a_4 := R_{2sc} \qquad a_4 = 2.34\,\Omega$

$L_{2sc} := 118 \cdot \mu\mathrm{H} \qquad b_4 := \omega \cdot L_{2sc} \qquad b_4 = 44.484952\,\Omega$

Impedances of the primary coil:

$Z_1 := R_1 + i \cdot \omega \cdot L_1 \qquad A := Z_1 \qquad Z_1 = 45 + 1.225221i \times 10^4\,\Omega$

$Z_3 := R_{1sc} + i \cdot \omega \cdot L_{1sc} \qquad Z_3 = 72 + 1.447646i \times 10^3\,\Omega$

Impedances of the secondary coil:

$Z_2 := R_2 + i \cdot \omega \cdot L_2 \qquad D := Z_2 \qquad Z_2 = 2.5 + 3.792531i \times 10^2\,\Omega$

$Z_4 := R_{2sc} + i \cdot \omega \cdot L_{2sc} \qquad Z_4 = 2.34 + 44.484952i\,\Omega$

In the following Chapters 4 to 7 several methods for the combined determination of R_m and M, derived from the above given resistance- and inductance measurement values, are presented.

Chapter 4

Application of the Complex Relative-Error Equalization Theory to Measurement Values of the Experimentation Transformer

4.1 The complex-geometrical mean of both impedance ratios (primary coil and secondary coil) of the experimentation transformer of Chapter 3:

$$\zeta_p := \frac{R_{1sc} + i \cdot \omega \cdot L_{1sc}}{R_1 + i \cdot \omega \cdot L_1} = \frac{Z_{1sc}}{Z_1} \qquad \zeta_s := \frac{R_{2sc} + i \cdot \omega \cdot L_{2sc}}{R_2 + i \cdot \omega \cdot L_2} = \frac{Z_{2sc}}{Z_2}$$

$$\zeta_p = 0.118174 - 5.442461i \times 10^{-3} \qquad \zeta_s = 0.117332 - 5.396583i \times 10^{-3}$$

The components of these impedance ratios can be calculated not only with Mathcad, as shown just, but also, of course, in an elementary way, as derived already in the upper part of page 37 of Chapter 2 :

$$\zeta_{pr} := \frac{a_1 \cdot a_3 + b_1 \cdot b_3}{a_1^{\,2} + b_1^{\,2}} \qquad \zeta_{pr} = 0.118174 \qquad \zeta_{sr} := \frac{a_2 \cdot a_4 + b_2 \cdot b_4}{a_2^{\,2} + b_2^{\,2}} \qquad \zeta_{sr} = 0.117332$$

$$\zeta_{pi} := \frac{a_1 \cdot b_3 - a_3 \cdot b_1}{a_1^{\,2} + b_1^{\,2}} \qquad \zeta_{pi} = -5.442461 \times 10^{-3} \qquad \zeta_{si} := \frac{a_2 \cdot b_4 - a_4 \cdot b_2}{a_2^{\,2} + b_2^{\,2}} \qquad \zeta_{si} = -5.396583 \times 10^{-3}$$

The corresponding polar angles are:

$$\alpha := \operatorname{atan}\left(\frac{\zeta_{pi}}{\zeta_{pr}}\right) + (\zeta_{pr} < 0) \cdot (\zeta_{pi} > 0) \cdot \pi + (\zeta_{pr} < 0) \cdot (\zeta_{pi} < 0) \cdot \pi + (\zeta_{pr} > 0) \cdot (\zeta_{pi} < 0) \cdot 2 \cdot \pi$$

$$\beta := \operatorname{atan}\left(\frac{\zeta_{si}}{\zeta_{sr}}\right) + (\zeta_{sr} < 0) \cdot (\zeta_{si} > 0) \cdot \pi + (\zeta_{sr} < 0) \cdot (\zeta_{si} < 0) \cdot \pi + (\zeta_{sr} > 0) \cdot (\zeta_{si} < 0) \cdot 2 \cdot \pi$$

The <u>abs</u>olute value ζ_{gabs} and the polar angle γ of the geometrical mean are according to page 37 :

$$\zeta_{gabs} := \sqrt[4]{\left(\zeta_{pr}^{\,2} + \zeta_{pi}^{\,2}\right) \cdot \left(\zeta_{sr}^{\,2} + \zeta_{si}^{\,2}\right)} \qquad \gamma := \frac{\alpha + \beta}{2} \qquad \gamma = 6.237193\,\text{rad}$$

$$a_g := \zeta_{gabs} \cdot \cos(\gamma) \qquad \underline{\underline{a_g = 0.117752}} \qquad b_g := \zeta_{gabs} \cdot \sin(\gamma) \qquad \underline{\underline{b_g = -5.419474 \times 10^{-3}}}$$

For a comparison the very simple calculation of the components a_a and b_a of the arithmetical mean according to the lower part of page 37 . The subscript a stands for arithmetical :

$$a_a := \frac{\zeta_{pr} + \zeta_{sr}}{2} \qquad a_a = 0.117753 \qquad b_a := \frac{\zeta_{pi} + \zeta_{si}}{2} \qquad b_a = -5.419522 \times 10^{-3}$$

But the application of the CREET will be continued with the complex-geometrical mean.
The subscript g stands for geometrical :

$$\zeta_g := a_g + i \cdot b_g$$

4.1p Equalization of the impedance measurement values of the primary side:

According to the CREET of Chapter 2 the following substitutions are introduced:

$$m := a_g \cdot \frac{a_1}{a_3} \qquad n := b_g \cdot \frac{b_1}{a_3} \qquad \mu := a_g \cdot \frac{b_1}{b_3} \qquad \nu := b_g \cdot \frac{a_1}{b_3}$$

$$a := m^2 + \nu^2 + 1 \qquad b := n^2 + \mu^2 + 1 \qquad c := 2 \cdot (\mu \cdot \nu - m \cdot n)$$

$$d := 2 \cdot \left(m^2 - m \cdot n - m + \nu^2 + \mu \cdot \nu - \nu\right) \qquad g := 2 \cdot \left(n^2 - m \cdot n + n + \mu^2 + \mu \cdot \nu - \mu\right)$$

$$h := m^2 + n^2 + 1 - 2 \cdot m \cdot n - 2 \cdot m + 2 \cdot n + \mu^2 + \nu^2 + 1 + 2 \cdot \mu \cdot \nu - 2 \cdot \mu - 2 \cdot \nu$$

In order to find the correctives α_1, β_1, α_3, β_3, the sum Φ_p of the squares of the four relative correctives, as written on page 50, must be a minimum. This condition is fulfilled, if the minimum of the polynomial function $\Psi_p(x,y)$, as written half below, is found.
According to the CREET the location of the minimum is determined by:

$$x := \frac{c \cdot g - 2 \cdot b \cdot d}{4 \cdot a \cdot b - c \cdot c} \qquad x = 1.298234 \times 10^{-4} \qquad y := \frac{c \cdot d - 2 \cdot a \cdot g}{4 \cdot a \cdot b - c \cdot c} \qquad y = 2.601708 \times 10^{-3}$$

And according CREET for the primary side the following complex postulate has to be fulfilled:

$$\zeta_g = a_g + i \cdot b_g = \frac{A_3 + i \cdot B_3}{A_1 + i \cdot B_1},$$ in which A_1, B_1, A_3, B_3 are equalized

measurement values. This complex equation can be decomposed into one real part equation and one imaginary part equation. By means of such two equations the sum Φ_p can be transformed into a polynomial function with the coefficients a, b, c, d, g and the addendum h, as shown on the page 41, but denoted here on this page with $\Psi_p(x,y)$ for a formal distinction from the definition of Φ_p.

$$\Psi_p(x,y) := a \cdot x^2 + b \cdot y^2 + c \cdot x \cdot y + d \cdot x + g \cdot y + h \qquad \Psi_p(x,y) = 1.085824 \times 10^{-5}$$

$$\alpha_1 := R_1 \cdot x \qquad \alpha_1 = 5.842054 \times 10^{-3}\,\Omega \qquad \beta_1 := \omega \cdot L_1 \cdot y \qquad \beta_1 = 31.87668\,\Omega$$

$$A_1 := R_1 + \alpha_1 \qquad A_1 = 45.005842\,\Omega \qquad B_1 := \omega \cdot L_1 + \beta_1 \qquad B_1 = 1.228409 \times 10^4\,\Omega$$

$$A_3 := a_g \cdot A_1 - b_g \cdot B_1 \qquad A_3 = 71.872829\,\Omega \qquad B_3 := a_g \cdot B_1 + b_g \cdot A_1 \qquad B_3 = 1.446233 \times 10^3\,\Omega$$

$$\alpha_3 := A_3 - R_{1sc} \qquad \alpha_3 = -0.127171\,\Omega \qquad \beta_3 := B_3 - \omega \cdot L_{1sc} \qquad \beta_3 = -1.413093\,\Omega$$

Numerical proof of the equality of $\Psi_p(x,y)$ with Φ_p:

$$\Phi_p := \left(\frac{\alpha_1}{R_1}\right)^2 + \left(\frac{\beta_1}{\omega \cdot L_1}\right)^2 + \left(\frac{\alpha_3}{R_{1sc}}\right)^2 + \left(\frac{\beta_3}{\omega \cdot L_{1sc}}\right)^2 \qquad \Phi_p = 1.085824 \times 10^{-5}$$

4.1s Equalization of the impedance measurement values of the secondary side:

In comparison with the subscripts 1 and 3 of the foregoing Section 4.1p here are applied the subscripts 2 and 4 , and both variables x and y are replaced by ξ and η .

$$m := a_g \cdot \frac{a_2}{a_4} \qquad n := b_g \cdot \frac{b_2}{a_4} \qquad \mu := a_g \cdot \frac{b_2}{b_4} \qquad \nu := b_g \cdot \frac{a_2}{b_4}$$

The parameters m , n , μ , ν are different from that used before and yield the new coefficients a , b , c , d , g and the new additive constant h of the polynomial function $\Psi_s(\xi,\eta)$ given below.

$$a := m^2 + \nu^2 + 1 \qquad b := n^2 + \mu^2 + 1 \qquad c := 2\cdot(\mu\cdot\nu - m\cdot n)$$

$$d := 2\cdot\left(m^2 - m\cdot n - m + \nu^2 + \mu\cdot\nu - \nu\right) \qquad g := 2\cdot\left(n^2 - m\cdot n + n + \mu^2 + \mu\cdot\nu - \mu\right)$$

$$h := m^2 + n^2 + 1 - 2\cdot m\cdot n - 2\cdot m + 2\cdot n + \mu^2 + \nu^2 + 1 + 2\cdot\mu\cdot\nu - 2\cdot\mu - 2\cdot\nu$$

In order to find the correctives α_2 , β_2 , α_4 , β_4 , the sum Φ_s of the squares of the four relative correctives, as written on page 50, must be a minimum. This condition is fulfilled, if the minimum of the polynomial function $\Psi_s(\xi,\eta)$, as written half below, is found.
According to the CREET the location of the minimum is determined by:

$$\xi := \frac{c\cdot g - 2\cdot b\cdot d}{4\cdot a\cdot b - c\cdot c} \qquad \xi = -2.320815\times 10^{-4} \qquad \eta := \frac{c\cdot d - 2\cdot a\cdot g}{4\cdot a\cdot b - c\cdot c} \qquad \eta = -2.598935\times 10^{-3}$$

And according CREET for the secondary side the following complex postulate has to be fulfilled:

$$\zeta_g = a_g + i\cdot b_g = \frac{A_4 + i\cdot B_4}{A_2 + i\cdot B_2} \text{ ,} \qquad \text{in which} \qquad A_2,\ B_2,\ A_4,\ B_4 \qquad \text{are equalized}$$

measurement values. This complex equation can be decomposed into one real part equation and one imaginary part equation. By means of such two equations the sum Φ_s can be transformed into a polynomial function with the coefficients a , b , c , d , g and the addendum h , as shown on the page 41, but denoted here analogously to $\Psi_p(x,y)$ with $\Psi_s(\xi,\eta)$ for a formal distinction from the definition of Φ_s .

$$\Psi_s(\xi,\eta) := a\cdot\xi^2 + b\cdot\eta^2 + c\cdot\xi\cdot\eta + d\cdot\xi + g\cdot\eta + h \qquad \Psi_s(\xi,\eta) = 1.116643\times 10^{-5}$$

$$\alpha_2 := R_2\cdot\xi \qquad \alpha_2 = -5.802036\times 10^{-4}\,\Omega \qquad \beta_2 := \omega\cdot L_2\cdot\eta \qquad \beta_2 = -0.985654\,\Omega$$

$$A_2 := R_2 + \alpha_2 \qquad A_2 = 2.49942\,\Omega \qquad B_2 := \omega\cdot L_2 + \beta_2 \qquad B_2 = 3.782674\times 10^{2}\,\Omega$$

$$A_4 := a_g\cdot A_2 - b_g\cdot B_2 \qquad A_4 = 2.344322\,\Omega \qquad B_4 := a_g\cdot B_2 + b_g\cdot A_2 \qquad B_4 = 44.528223\,\Omega$$

$$\alpha_4 := A_4 - R_{2sc} \qquad \alpha_4 = 4.322327\times 10^{-3}\,\Omega \qquad \beta_4 := B_4 - \omega\cdot L_{2sc} \qquad \beta_4 = 4.327059\times 10^{-2}\,\Omega$$

Numerical proof of the equality of $\Psi_s(\xi,\eta)$ with Φ_s :

$$\Phi_s := \left(\frac{\alpha_2}{R_2}\right)^2 + \left(\frac{\beta_2}{\omega\cdot L_2}\right)^2 + \left(\frac{\alpha_4}{R_{2sc}}\right)^2 + \left(\frac{\beta_4}{\omega\cdot L_{2sc}}\right)^2 \qquad \Phi_s = 1.116643\times 10^{-5}$$

4.2 Comparison of the original with the equalized measurement values:

The subscript e stands for the equalized measurement values, which are equalized by means of the **CREET.** On the right-hand side of the following table are written the calculated relative correctives,

$R_1 = 45\Omega$	$R_{1e} := A_1$	$R_{1e} = 45.005842\Omega$	$C_1 := \frac{\alpha_1}{a_1}$	$C_1 = 1.298234 \times 10^{-4}$
$L_1 = 32.5\,mH$	$L_{1e} := \frac{B_1}{\omega}$	$L_{1e} = 32.584556\,mH$	$C_2 := \frac{\beta_1}{b_1}$	$C_2 = 2.601708 \times 10^{-3}$
$R_{1sc} = 72\Omega$	$R_{1sce} := A_3$	$R_{1sce} = 71.872829\Omega$	$C_3 := \frac{\alpha_3}{a_3}$	$C_3 = -1.766259 \times 10^{-3}$
$L_{1sc} = 3.84\,mH$	$L_{1sce} := \frac{B_3}{\omega}$	$L_{1sce} = 3.836252\,mH$	$C_4 := \frac{\beta_3}{b_3}$	$C_4 = -9.761313 \times 10^{-4}$
$R_2 = 2.5\Omega$	$R_{2e} := A_2$	$R_{2e} = 2.49942\Omega$	$C_5 := \frac{\alpha_2}{a_2}$	$C_5 = -2.320815 \times 10^{-4}$
$L_2 = 1.006\,mH$	$L_{2e} := \frac{B_2}{\omega}$	$L_{2e} = 1.003385\,mH$	$C_6 := \frac{\beta_2}{b_2}$	$C_6 = -2.598935 \times 10^{-3}$
$R_{2sc} = 2.34\Omega$	$R_{2sce} := A_4$	$R_{2sce} = 2.344322\Omega$	$C_7 := \frac{\alpha_4}{a_4}$	$C_7 = 1.847148 \times 10^{-3}$
$L_{2sc} = 1.18 \times 10^2\,\mu H$	$L_{2sce} := \frac{B_4}{\omega}$	$L_{2sce} = 1.181148 \times 10^2\,\mu H$	$C_8 := \frac{\beta_4}{b_4}$	$C_8 = 9.727016 \times 10^{-4}$

4.3 The Standard Corrective Sigma = σ according to page 43 of Chapter 2 : Pythagorean Mean

$$\text{Sigma} = \sigma := \sqrt{\frac{C_1{}^2 + C_2{}^2 + C_3{}^2 + C_4{}^2 + C_5{}^2 + C_6{}^2 + C_7{}^2 + C_8{}^2}{8}}$$

With $\Theta = 1 + i \cdot 0$ is $\sigma = 1.659242 \times 10^{-3}$ $\sigma^2 = 2.753084 \times 10^{-6}$
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(But with $\Theta = 0.999905 - i \cdot 3.3 \cdot 10^{-6}$ of page 47 would be $\sigma = 1.658895 \cdot 10^{-3}$, i.e. only $0.2\,‰$ smaller.)

The arithmetical mean of all eight individual relative correctives is :

$$C_{am} := \frac{1}{8} \cdot (C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8) \qquad C_{am} = -2.753084 \times 10^{-6}$$
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Surprisingly C_{am} is oppositely equal to the square of Sigma = σ. Two numerical verifications and the algebraical proof are given in 3 Appendices on the pages 124 to 131.

$$C_{amn} := \frac{C_{am}}{\sigma} \qquad C_{amn} = -1.659242 \times 10^{-3} \qquad D_{rel} = 1 + \frac{C_{am}}{\sigma^2} = 2.174927 \times 10^{-12}$$
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C_{amn} is the normalized C_{am}, and D_{rel} represents the relative difference between C_{am} and σ^2.
See also the pages 118 and 119.

4.4 Calculations of k_g, B_e, R_{me}, M_e, k_{proxe}, k_{inde} :

$Z_{1sce} := R_{1sce} + i \cdot \omega \cdot L_{1sce}$ $\qquad$ $Z_{2sce} := R_{2sce} + i \cdot \omega \cdot L_{2sce}$

$Z_{1sce} = 71.872829 + 1.446233i \times 10^3\,\Omega$ $\qquad$ $Z_{2sce} = 2.344322 + 44.528223i\,\Omega$

$Z_{1e} := R_{1e} + i \cdot \omega \cdot L_{1e}$ $\quad A_e := Z_{1e}$ $\qquad$ $Z_{2e} := R_{2e} + i \cdot \omega \cdot L_{2e}$ $\quad D_e := Z_{2e}$

$A_e = 45.005842 + 1.228409i \times 10^4\,\Omega$ $\qquad$ $D_e = 2.49942 + 3.782674i \times 10^2\,\Omega$

$$\zeta_{pe} := \frac{Z_{1sce}}{Z_{1e}} \qquad \zeta_{se} := \frac{Z_{2sce}}{Z_{2e}}$$

$\zeta_{pe} = 0.117752 - 5.419474i \times 10^{-3}$ $\qquad$ $\zeta_{se} = 0.117752 - 5.419474i \times 10^{-3}$

Because ζ_{pe} and ζ_{se} are equalized ratios, they are equal to each other, and the **complex-geometrical mean** ζ_g is equal to both. Concerning the numerical example see also page 46 :

$$\zeta_g := \sqrt{\zeta_p \cdot \zeta_s} \qquad \zeta_g = 0.117752 - 5.419474i \times 10^{-3}$$

In order to calculate the complex coupling coefficient, one has to refer to the square of them, explained in connection with the complex transformation ratio in Chapter 8 :

$$k^2 = \frac{C \cdot B}{A \cdot D}$$

Additionally is repeated from the pages 22 and 23 of Chapter 1 :

$$B \cdot C = A \cdot D \cdot \left(1 - \frac{Z_{1sc}}{Z_1}\right) \quad \text{and} \quad C \cdot B = A \cdot D \cdot \left(1 - \frac{Z_{2sc}}{Z_2}\right) \quad \text{and} \quad C^2 = B^2$$

And from Chapter 2 :

$$\zeta_p = \frac{Z_{1sc}}{Z_1} \quad \text{and} \quad \zeta_s = \frac{Z_{2sc}}{Z_2} \quad \text{and} \quad \zeta_g = \sqrt{\zeta_p \cdot \zeta_s}$$

These equations yield directly $\qquad k_g^2 = 1 - \zeta_g$

The square of the complex coupling coefficient k_g is the radicand

$$1 - \zeta_g = 0.882248 + 5.419474i \times 10^{-3}$$

of the following square root. The accidentally correct sign of this square root is accepted in such a way that the real part of the complex coupling coefficient is positive:

$$k_g := \sqrt{1 - \zeta_g} \qquad k_g = 0.939285 + 2.884894i \times 10^{-3}$$

But in order to solve the sign question without this statement, the complex coupling coefficient k_g should be calculated by means of polar coordinates:

Chapter 4.4 Equalized Determination of B_e

With the set-up of $k_g{}^2 = 1 - \mathrm{Re}(\zeta_g) - i\cdot\mathrm{Im}(\zeta_g) = p + i\cdot q$

are the real part p and the imaginary part q of the square of the coupling coefficient k_g :

$$p := 1 - \mathrm{Re}(\zeta_g) \qquad q := -\mathrm{Im}(\zeta_g)$$

$$p = 0.882248 \qquad q = 5.419474 \times 10^{-3}$$

The angle between the vector p + i q and the positive direction of the abscissa is counted in the counterclockwise sense, that is mathematically the positive sense. Hence applies to all 4 quadrants:

$$\phi_g := \mathrm{atan}\left(\frac{q}{p}\right) + (p<0)\cdot(q>0)\cdot\pi + (p<0)\cdot(q<0)\cdot\pi + (p>0)\cdot(q<0)\cdot2\cdot\pi$$

$$\phi_g = 6.142725 \times 10^{-3}\,\mathrm{rad} \qquad \phi_g = 0.351952\,\mathrm{Grad}$$

The absolut value is the hypothenuse: $h_g := \sqrt{p^2 + q^2}$ $h_g = 0.882265$

Consequently the complex coupling coefficient is: $k_g = k_{real} + i\cdot k_{im}$

$$k_g := \sqrt{h_g}\cdot e^{i\cdot\frac{\phi_g}{2}} \qquad k_g = 0.939285 + 2.884894i \times 10^{-3}$$

in accordance with the foregoing direct square root calculation. But with other devices under test the positive sign of the square root in the foregoing calculation would not exists always necessarily. Hints and explanations are given among others on the pages 32 and 34.

According to the equation in the upper part of page 24, it follows in connection with $B = C$ of page 23 the mutual coefficient B by extracting still the square root. Therefore it can be said that B_e is equal to the product of k_g and the complex-geometrical mean of the equalized coefficients A_e and D_e .

$$B_e := k_g\cdot\sqrt{A_e\cdot D_e} \qquad B_e = 4.179597 + 2.02477i \times 10^3\,\Omega$$

In order to determine the equalized mutual impedance B_e unambiguously for a whatever two-port network, here now both complex coefficients A_e and D_e are transformed into polar coordinates:

$$A_{er} := \mathrm{Re}(A_e) \qquad A_{ei} := \mathrm{Im}(A_e) \qquad D_{er} := \mathrm{Re}(D_e) \qquad D_{ei} := \mathrm{Im}(D_e)$$

$$A_{er} = 45.005842\,\Omega \qquad A_{ei} = 1.228409 \times 10^4\,\Omega \qquad D_{er} = 2.49942\,\Omega \qquad D_{ei} = 3.782674 \times 10^2\,\Omega$$

$$\alpha_e := \mathrm{atan}\left(\frac{A_{ei}}{A_{er}}\right) + (A_{er}<0)\cdot(A_{ei}>0)\cdot\pi + (A_{er}<0)\cdot(A_{ei}<0)\cdot\pi + (A_{er}>0)\cdot(A_{ei}<0)\cdot2\cdot\pi$$

$$\delta_e := \mathrm{atan}\left(\frac{D_{ei}}{D_{er}}\right) + (D_{er}<0)\cdot(D_{ei}>0)\cdot\pi + (D_{er}<0)\cdot(D_{ei}<0)\cdot\pi + (D_{er}>0)\cdot(D_{ei}<0)\cdot2\cdot\pi$$

$$\alpha_e = 1.567133\,\mathrm{rad} \qquad \alpha_e = 89.790083\,\mathrm{Grad} \qquad \delta_e = 1.564189\,\mathrm{rad} \qquad \delta_e = 89.621421\,\mathrm{Grad}$$

$$R_{ae} := \sqrt{A_{er}{}^2 + A_{ei}{}^2} \qquad R_{de} := \sqrt{D_{er}{}^2 + D_{ei}{}^2}$$

$$R_{ae} = 1.228417 \times 10^4\,\Omega \qquad R_{de} = 3.782757 \times 10^2\,\Omega$$

Coherent Determination of R_{me}, M_e, k_{proxe} and k_{inde} Chapter 4.4

$$B_e = \sqrt{h_g}\cdot e^{i\cdot\frac{\phi_g}{2}}\cdot\sqrt{R_{ae}\cdot R_{de}}\cdot e^{i\cdot\frac{\alpha_e+\delta_e}{2}} \qquad B_e := \sqrt{h_g\cdot R_{ae}\cdot R_{de}}\cdot e^{i\cdot\left(\frac{\phi_g+\alpha_e+\delta_e}{2}\right)}$$

$$B_e = 4.179597 + 2.02477i \times 10^{3}\,\Omega$$

This result agrees exactly with that of the foregoing page 52.

The real part of the equalized mutual impedance B_e is the equalized mutual internal resistance R_{me} of the device under test, which is here the electromagnetic transformer of Chapter 3 .
And the imaginary part, divided by the angular frequency ω , is the equalized mutual inductance M_e .

$$R_{me} := \mathrm{Re}(B_e) \qquad M_e := \frac{\mathrm{Im}(B_e)}{\omega}$$

$$R_{me} = 4.179597\,\Omega \qquad M_e = 5.370868\,\mathrm{mH}$$

============== ==============

These are the best possible equalized values thanks to the developed CREET. This calculation demonstrates that the mutual resistance R_{me} is defined as well as the mutual inductance M_e , therefore the title of my subject "Coherent Mathematical Treatment of the Inductive and Resistive Coupling Acting in Electromagnetic Transformers".

The equalized values will be used in the following Chapters as reference values for comparisons with the results of other methods, where unequalized original measurement values are applied. The definition of the resistive proximity-coupling coefficient and that of the inductive coupling coefficient are given already in the introduction and in the Chapter 1, and these definitions may be repeated here again, but now applied to the equalized measurement values of this Chapter 4.2 :

$$k_{proxe} := \frac{R_{me}}{\sqrt{R_{1e}\cdot R_{2e}}} \qquad k_{proxe} = 0.394076 \qquad k_{inde} := \frac{M_e}{\sqrt{L_{1e}\cdot L_{2e}}} \qquad k_{inde} = 0.939301$$

============== =============

k_{proxe} is the equalized resistive Proximity-Effect coupling coefficient.

k_{inde} is the equalized inductive coupling coefficient.

The subscript e stands again for equalized. Also it has to be said that the reason for the presented additional calculation in polar coordinates comprises the intention that the CREET can be used also for other two-port networks and not only just for transformers consisting of inductances and resistances, but also for other two-port networks, which could contain also capacitances.
In Chapter 13 are investigated two simple LR-Π-circuits. And in order to solve the sign problem of the mutual impedance B , polar coordinates are applied there, of course.

If the mutual impedance $B_e = R_{me} + i\cdot\omega\cdot M_e$ would be determined for instance by means of the Two-Path Method of Chapter 7, applied to equalized measurement values of the page 50 then the complex coupling coefficient k_e is exactly equal to the directly calculated k_g on the foregoing page, because of the equality of the ζ_g -definitions on page 35 half below and page 38 half way up.

$$k_e := \frac{B_e}{\sqrt{A_e\cdot D_e}} \qquad k_e = 0.939285 + 2.884894i \times 10^{-3} \qquad k_g = 0.939285 + 2.884894i \times 10^{-3}$$

The components of the complex coupling coefficient $k_e = k_g$ are:

$$k_{real} := \mathrm{Re}(k_g) \qquad k_{real} = 0.939285 \qquad k_{im} := \mathrm{Im}(k_g) \qquad k_{im} = 2.884894 \times 10^{-3}$$

The equalization subscript e is omitted in the last line, because the complex coupling coefficient depends directly from the geometrical mean ζ_g and is therefore independent upon the whole equalization process.

10.4 An example for the calculation of C_{opt}, R_{opt} and η_{max} :

For these calculations are used the measurement values equalized in Chapter 4 . These values are based on the open-circuit- and the short-circuit impedance measurement values of a self-wound transformer, described in Chapter 3 and measured with an impedance analyser of Hewlett & Packard.

The used measurement frequency is: $f := 60 \cdot kHz$ $\omega := 2 \cdot \pi \cdot f$

The mutual inductive coupling coefficient and the mutual Proximity-Effect coupling coefficient are:

$$k_{inde} := \frac{M_e}{\sqrt{L_{1e} \cdot L_{2e}}} \qquad k_{inde} = 0.939301 \qquad k_{proxe} := \frac{R_{me}}{\sqrt{R_{1e} \cdot R_{2e}}} \qquad k_{proxe} = 0.394076$$

The subscript e stands for equalized.
The quality factor of the primary coil and that of the secondary coil are:

$$Q_{1e} := \frac{\omega \cdot L_{1e}}{R_{1e}} \qquad Q_{1e} = 2.729443 \times 10^2 \qquad Q_{2e} := \frac{\omega \cdot L_{2e}}{R_{2e}} \qquad Q_{2e} = 1.513421 \times 10^2$$

The optimum terminating capacitance is:

$$C_{opte} := \frac{1 - k_{inde} \cdot k_{proxe} \cdot \sqrt{\frac{Q_{1e}}{Q_{2e}}}}{\left(k_{inde} \cdot \sqrt{\frac{Q_{1e}}{Q_{2e}}} - k_{proxe}\right)^2 + \left(1 - k_{proxe}^2\right) \cdot \left(1 + \frac{1}{Q_{2e}^2}\right)} \cdot \frac{1}{\omega^2 \cdot L_{2e}}$$

$$C_{opte} = 2.208199\,nF$$
===============

The optimum terminating resistance is:

$$R_{opte} := \frac{\left(k_{inde} \cdot \sqrt{\frac{Q_{1e}}{Q_{2e}}} - k_{proxe}\right)^2 + \left(1 - k_{proxe}^2\right) \cdot \left(1 + \frac{1}{Q_{2e}^2}\right)}{\sqrt{\left(1 + k_{inde}^2 \cdot Q_{1e} \cdot Q_{2e}\right) \cdot \left(1 - k_{proxe}^2\right)}} \cdot Q_{2e} \cdot \omega \cdot L_{2e}$$

$$R_{opte} = 0.521067\,k\Omega$$
================

The maximum possible power transfer efficiency is:

$$\eta_{maxe} := \frac{k_{inde}^2 \cdot Q_{1e} \cdot Q_{2e} + k_{proxe}^2}{\left(\sqrt{1 + k_{inde}^2 \cdot Q_{1e} \cdot Q_{2e}} + \sqrt{1 - k_{proxe}^2}\right)^2}$$

$$\eta_{maxe} = 0.990418$$
==============

10.5 A verification of the analytically derived η_{max} -formula:

Additionally to the performed analytical derivation, in the following is given still a verification, which properly should not be needed. Therefore it should be seen only as an interest addendum. Let oss assume an effective input voltage of e. g. U_1 = 1 V sine. This is about 200 times smaller than a possible working voltage. The assumed small input voltage is in accordance with that transformer impedances given in Chapter 3 , which were measured at small signal amplitudes. The positive direction of the secondary current is here defined in opposite to the foregoing Chapters as the direction from the upper end of the secondary coil into the terminating impedance, which for the sake of simplicity may consits here of the just calculated parallel connection of C_{opte} and R_{opte}.

In order to calculate the magnetic flux density B_{peak} at the RMS input voltage U_1, also the effective cross section area A_{eff} of the ferrite core, the primary turn number N_1 of the primary coil and the frequency f have to be taken into consideration:

$A_{eff} := 21.6\cdot mm^2$ $\quad U_1 := 1\cdot V$ $\quad N_1 := 400$ $\quad f := 60\cdot kHz$

$$B_{peak} := \sqrt{2}\cdot\frac{U_1}{N_1\cdot 2\cdot\pi\cdot f\cdot A_{eff}} \qquad B_{peak} = 4.341803\times 10^{-4}\,T$$

For a comparison with similar cases it may be mentioned that measurements of the initial permeability usually are carried out at a magnetic flux density with an amplitude of 0.1 mT (Philips) or 0.25 mT (Siemens). T = Tesla = Vs / m^2. (Databooks are mentioned on page 3.)
For the following calculation the equalized measurement values and the optimized load impedance are used here. It is assumed that the capacitor with the capacitance C_{opte} is completely loss-free.

$R_{1e} = 45.005842\,\Omega$ $\quad R_{2e} = 2.49942\,\Omega$ $\quad R_{me} = 4.179597\,\Omega$ $\quad R_{opte} = 0.521067\,k\Omega$

$L_{1e} = 32.584556\,mH$ $\quad L_{2e} = 1.003385\,mH$ $\quad M_e = 5.370868\,mH$ $\quad C_{opte} = 2.208199\,nF$

$$Z_{opte} := \frac{1}{\frac{1}{R_{opte}} + i\cdot\omega\cdot C_{opte}} \qquad Z_{opte} = 4.385498\times 10^2 - 1.902313i\times 10^2\,\Omega$$

Optimized terminating impedance.

From the 2nd two-port equation of page 70 follows with respect to equalized values:

$$I_1 = \frac{Z_{opte} + i\cdot\omega\cdot L_{2e} + R_{2e}}{i\cdot\omega\cdot M_e + R_{me}}\cdot I_2$$

Insertion into the 1st two-port equation yields:

$$I_2 := \frac{U_1}{\left[\frac{(i\cdot\omega\cdot L_{1e} + R_{1e})\cdot(Z_{opte} + i\cdot\omega\cdot L_{2e} + R_{2e})}{i\cdot\omega\cdot M_e + R_{me}} - i\cdot\omega\cdot M_e - R_{me}\right]}$$

$I_2 = 0.336862 + 0.111921i\,mA$

$\arg(I_2) = 0.320773\,rad$

$$U_2 := Z_{opte}\cdot I_2 \qquad U_2 = 0.169022 - 1.499859i\times 10^{-2}\,V$$

$$P_{out} := \frac{(|U_2|)^2}{R_{opte}}$$ Dissipated output power $\quad P_{out} = 5.525827\times 10^{-5}\,W$

$$I_1 := \frac{Z_{opte} + i\cdot\omega\cdot L_{2e} + R_{2e}}{i\cdot\omega\cdot M_e + R_{me}}\cdot I_2 \qquad I_1 = 5.579289\times 10^{-2} - 6.286849i\times 10^{-2}\,mA$$

$\arg(I_1) = -0.844956\,rad$

Chapter 10.5 A verification of the η_{max}-Formula:

$\alpha := \arg(I_1) - \arg(U_1)$ with $\arg(U_1) = 0$ phase shift $\alpha = -0.844956\,\mathrm{rad}$

$P_1 := |I_1| \cdot |U_1| \cdot \cos(\alpha)$ Input power $P_1 = 5.579289 \times 10^{-5}\,\mathrm{W}$

The power transfer efficiency in this verification and its neglectable small difference to η_{maxe} are:

$\eta_{ver1} := \dfrac{P_{out}}{P_1}$ $\eta_{ver1} = 0.990418$ $\dfrac{\eta_{ver1}}{\eta_{maxe}} - 1 = -2.220446 \times 10^{-16}$

A samewhat modified calculation of the output power, called now P_2, is:

$\beta := \arg(I_2) - \arg(U_2)$ phase shift $\beta = 0.409278\,\mathrm{rad}$

$P_2 := |I_2| \cdot |U_2| \cdot \cos(\beta)$ Output power $P_2 = 5.525827 \times 10^{-5}\,\mathrm{W}$

$\eta_{ver2} := \dfrac{P_2}{P_1}$ $\eta_{ver2} = 0.990418$ $\dfrac{\eta_{ver2}}{\eta_{maxe}} - 1 = -1.110223 \times 10^{-16}$

The power dissipation of the transformer in this small signal application is only : $\delta := P_1 - P_2$ $\delta = 5.346232 \times 10^{-7}\,\mathrm{W}$

An other way for the determination of the total power dissipation is the calculation of the **individual dissipations** D_1 and D_2 and the **mutual power generation** G_m :

$D_1 := R_{1e} \cdot \left(|I_1|\right)^2$ $D_2 := R_{2e} \cdot \left(|I_2|\right)^2$ $G_m := 2 \cdot R_{me} \cdot |I_1| \cdot |I_2| \cdot \cos\left(\arg(I_2) - \arg(I_1) + \pi\right)$

$D_1 = 3.179795 \times 10^{-7}\,\mathrm{W}$ $D_2 = 3.149325 \times 10^{-7}\,\mathrm{W}$ $G_m = -9.828883 \times 10^{-8}\,\mathrm{W}$

The two power dissipations D_1 and D_2 and the mutual power generatiion G_m are derived from the two two-port equations in such a way that the first equation, which determines the primary voltage U_1, is multiplied by the sine-shaped primary current I_1, and the second equation, which determines the secondary voltage U_2, is multiplied by the phase-shifted, but also sine-shaped secondary current I_2. In order to obtain the right sign of the generated power G_m, it has to be taken care about the normal functionality of a transformer, which means that the primary current I_1 flows into the transformer, but the secondary current I_2 out of them. Therefore is added a phase shift of π in the equation above:

The described derivation of the **mutual power dissipation yields a negative sign**, which means **a real generation of power**. Therefore the symbol G_m is used.

$S := D_1 + D_2 + G_m$ $S = 5.346232 \times 10^{-7}\,\mathrm{W}$ $D_1 + D_2 = 6.32912 \times 10^{-7}\,\mathrm{W}$

S is equal to the above calculated $\delta = 5.346232 \times 10^{-7}\,\mathrm{W}$

All three calculation methods harmonize exactly with one another.

$\eta_{ver3} := \dfrac{P_1 - S}{P_1}$ $\eta_{ver3} = 0.990418$ ============= $\dfrac{\eta_{ver3}}{\eta_{maxe}} - 1 = 2.220446 \times 10^{-16}$

The relative difference between η_{ver3} and η_{maxe} is here also below the very small residual calculation error of the PC, it means practically equal to zero. - ***Quod erat demonstrandum.***